International Journal of Theoretical Physics, Vol. 40, No. 10, October 2001 (© 2001)

Rieffel's Deformation Quantization and Isospectral Deformations

Andrzej Sitarz[1]

We demonstrate the relation between the isospectral deformation and Rieffel's deformation quantization by the action of $\mathbb{R}^n$.

1. INTRODUCTION

Isospectral deformation was introduced by Connes and Landi (2000), (see also Connes (2000)), with the detailed examples of the noncommutative 3 and 4 spheres, providing an interesting class of objects that generalize manifolds in the algebraic framework. It is worth noticing that similar examples, motivated by physics, though in a different context, were constructed in the sixties by Grossman *et al.* (1969) or, more recently, by Kulish and Mudrov (1999).

Isospectral deformation has been recently a subject of much interest (Connes and Dubois-Violette, in preparation). In particular, the symmetries of isospectral geometries (seen as Hopf algebras acting on the deformed algebras) appear to be a twist by a Cartan subalgebra of the universal enveloping algebra of Lie algebras, which were symmetries of the undeformed algebras (Sitarz, 2001); the constructed spectral triples are symmetric in the sense of Paschke and Sitarz (2000).

The analysis of symmetries and the generalized construction of the deformation, as well as the similarity of the construction with the well-known noncommutative torus has led to the connections between the isospectral deformations and Rieffel's deformation quantization by the action of $\mathbb{R}^n$ (Rieffel, 1993). In fact, both the general construction as well as the corresponding symmetries were already defined in a general setup by Rieffel (1993, 1994, 1995).

In this paper we show that the isospectral deformations as defined in Connes and Landi (2000) are special case of Rieffel's construction (Rieffel, 1993).

[1] Laboratoire de Physique Theorique, Université Paris-Sud, Bat. 210, 91405 Orsay Cedex, France; e-mail: andrzej.sitarz@th.u-psud.fr.

2. THE ISOSPECTRAL DEFORMATIONS

In the original article (Connes and Landi, 2000) one starts with the algebra $\mathcal{A}$ of smooth functions on a manifold, with an isometry group of rank $r \geq 2$. This is equivalent to the statement that the torus T^2 is the isometry subgroup of the algebra. Now, taking the elements t, which are of C^∞ class with respect to the action of the torus group $t \mapsto \alpha_s(t)$, one can obtain their decomposition as a norm convergent sum of homogeneous elements of a given bidegree. Then, for any homogeneous element one may define a deformed product (by a left or right twist). By linearity this extends to the linear combinations of homogeneous elements that are dense in the algebra we have started with.

To make correspondence with the Rieffel's deformation quantization we shall use the dual picture of isometries as used in Sitarz (2001).

Remark 1. Let $\mathcal{A}$ be an algebra and the torus T^2 be a subgroup of automorphisms of $\mathcal{A}$. Then for every operator T, which is of class C^∞ relative to the isometry α_s, the additive group $\mathbb{R}^2$ acts as follows:

$$[x_1, x_2] \triangleright t = \alpha_{(e^{2\pi i x_1}, e^{2\pi i x_2})}(t), \tag{1}$$

Using the generators of torus symmetries p_1, p_2 for instance

$$p_1 \triangleright T = \frac{1}{2\pi i}\left(\frac{d}{dx_1}([x_1, x_2] \triangleright t)\right)_{|x=0},$$

the relation (1) could be rewritten as

$$[x_1, x_2] \triangleright t = e^{2\pi i(x_1 p_1 + x_2 p_2)} \triangleright t.$$

Remark 2. With respect to the action of p_1, p_2 the operators, which are homogeneous of degree (n_1, n_2), behave like

$$p_1 \triangleright t = n_1 t, \qquad p_2 \triangleright t = n_2 t. \tag{2}$$

Remark 3. The product in the algebra $\mathcal{A}$ can be deformed, first on elements of given degree, and then extending the deformation by linearity:

$$a * b = ab\lambda^{n_1^a n_2^b}, \tag{3}$$

where λ is a complex number such that $|\lambda| = 1$. This gives the right twist of Connes and Landi (2000). For future reference we shall denote the deformed algebra by $\mathcal{A}_\lambda$.

It would be useful to introduce a *quantization map*:

$$\mathcal{A} \ni a \mapsto \underline{a} \in \mathcal{A}_\lambda,$$

so that the Eq. (3) could be rewritten as

$$\underline{a} * \underline{b} = \underline{ab}\lambda^{n_1^a n_2^b}. \tag{4}$$

This deformation could be also rewritten in a more abstract form as arising from the twist of the symmetry algebra by a Cartan element (Sitarz, 2001). Detailed analysis of the properties of such deformation are in Connes and Dubois-Violette (in preparation).

3. RIEFFEL'S DEFORMATION BY THE ACTION OF $\mathbb{R}^n$

3.1. General Construction

Suppose we have an algebra A and the action of $V = R^n$ on this algebra, and a linear map J from V', the dual of V to V such that it is skew-symmetric $J^T = -J$. Again, one takes a subalgebra of elements that are C^∞ vectors in A for the action of $\mathbb{R}^n$.

To make a direct correspondence with the above case of isospectral deformations we restrict ourselves to $n = 2$. In analogy with Poisson brackets on the function on a manifold we might define a Poisson bracket:

$$P(a, b) = \sum_i \alpha_{Jr_i}(a)\alpha_{p_i}(b), \tag{5}$$

where p_i is the basis of $\mathbb{R}^2$ and r_i of its dual. Clearly this is independent of the choice of the basis (see Rieffel, 1993, 1994) and makes the algebra A^∞ a strict Poisson* algebra.

Then using the oscillatory integrals one can define a deformed product:

$$a \times_J b = \int_V \int_{V'} \alpha_{Jy}(a)\alpha_x(b)e^{2\pi i(y\cdot x)}, \tag{6}$$

which could be recognized as a deformation quantization in the direction of the Poisson structure as defined in (5).

3.2. Equivalence With Isospectral Deformations

To obtain the deformation as defined in (3) we have to modify the expression (6) by allowing arbitrary (not necessarily antisymmetric) operator J.

Let us calculate explicitly for homogeneous elements a and b. We parameterize V (coordinates x, basis p_i) and V' (coordinates y, basis e_i), and with a particular choice of the map J.

$$Je_1 = 0, \qquad Je_2 = \theta p_1.$$

Then,

$$a \times_J b = \int_V \int_{V'} d^2x d^2y e^{2\pi i(\theta y_2 n_1^a)} a e^{2\pi i(x_1 n_1^b + x_2 n_2^b)} b e^{2\pi i(y_1 x_1 + y_2 x_2)} = \dots$$

Calculating further using the standard properties of oscillatory integrals, we obtain

$$\cdots = ab \int_V d^2x \int_{V'} d^2y e^{2\pi i(\theta n_1^a y_2 + n_2^b x_2 + y_2 x_2)} = abe^{2\pi i \theta n_1^a n_2^b},$$

which agrees with the definition for the right twist as defined in (3) with $\lambda = e^{2\pi i\theta}$.

4. CONCLUSIONS

Having shown the relation one might use the results valid for the Rieffel's deformation quantization to the case of isospectral deformations.

Especially interesting for physical applications might be results for deformations of noncompact groups (Rieffel, 1995) and the possible models using the twisted Minkowski space as in Kulish and Mudrov (1999).

ACKNOWLEDGMENTS

While finishing this paper we have learned of a similar work by Joseph Varilly (2001). The author also thanks Michel Dubois-Violette, J. M.Gracia-Bondía, Gianni Landi, John Madore, Mario Pachke, and Harold Steinacker for helpful discussions on many topics.

REFERENCES

Connes, A. (2000). Noncommutative Geometry Year 2000. arXiv:math. QA/0011193.

Connes, A. and Landi, G. (2000). Noncommutative manifolds, the instanton algebra and isospectral deformations. arXiv:math. QA/0011194.

Connes, A. and Dubois-Violette, M. Noncommutative finite-dimensional manifolds I. Spherical manifolds and related examples arXiv:math. QA/0107070.

Grossman, A., Loupias, G., and Stein, E. M. (1969). An algebra of pseudodifferential operators and quantum mechanics in phase space, *Ann. Inst. Fourier* **18**(2), 343–368.

Kulish, P. P. and Murdov, A. I. (1999). Twist-like geometries on a quantum Minkowski space. *Tr. Mat. Inst. Steklova* **226**, *Mat. Fiz. Probl. Kvantovoi Teor. Polya.* 97–111. On the occasion of the 65th birthday of Academician Lyudvig Dmitrievich Faddeev.

Paschke, M. and Sitarz, A. (2000). The geometry of noncommutative symmetries. *Acta Physica Polonica B* **31**(11), 1897–1911.

Rieffel, M. (1993). *Deformation Quantization for Actions of R^d, Vol. 506,* AMS, Providence. Memoirs AMS.

Rieffel, M. (1994). Quantization and C* Algebras. In *C* Algebras, 1943–1993, Contemporary Mathematics 167,* R. E. Dolan, ed., AMS, Providence.

Rieffel, M. (1995). Non-compact quantum groups associated with Abelian subgroups. *Communications in Mathematical Physics* **171**, 181.

Rieffel, M. (1993). *Deformation Quantization for Actions of R^d, Vol. 506,* AMS, Providence. Memoirs AMS.

Sitarz, A. (2001). Twists and spectral triples for isospectral deformations. arXiv:math. QA/0102074.

Várilly, J. C. (2001). Quantum symmetry groups of noncommutative spheres. arXiv:math. QA/0102065.